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# Heterogeneous elastic solids: a mixed homogenization-rigidification technique

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**Abstract**

A homogenization technique for heterogeneous elastic solids made up by a matrix containing inclusions modelled as *rigid in the limit*, is proposed. It is shown that the approach can cause considerable simplifications with respect to the use of standard homogenization procedures. The case of a masonry panel set across on an opening is analysed by applying the proposed technique and some numerical results are given. They are compared first with those obtained by considering the panel as a heterogeneous body and, in turn, by using standard homogenization. © 2000 Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

Periodic heterogeneous bodies that are made up by the repetition of a small cell, may be analysed as if they were homogeneous by using a homogenization technique. The core of the method is the formulation of a differential problem based on a single cell. The solution of such a problem leads to the definition of homogenized constitutive relations, which then work as macroscopic descriptors for the behaviour of a heterogeneous body.

Dealing with the differential problem on the cell can be in many cases a very cumbersome matter, simpler only in the case of a two-phase medium where one phase is modelled as a rigid body, with the drawback of the stress field being indeterminate there.

The aim of this article is to show that, by using a concept introduced by G. Grioli in the 1980s and modelling the inclusion as a *rigid in the limit* material, one keeps the advantages of the rigid modelling while eliminating its drawbacks.

Following this procedure, one deals with a problem whose complexity is of the same order as in the case of the rigid modelling while the stress field is no more indeterminate.

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The proposed procedure is applied to the analysis of a masonry panel set across on an opening. The body is modelled as a 2D continuum made up of two constituents: the mortar and the bricks. The former is described as a linear elastic material, the latter as *rigid in the limit*.

The same problem has been analysed with the bricks modelled as linear elastic bodies, as well, using a standard homogenization technique. The results obtained for two different stiffness ratios are compared with the previous ones.

## 2. Homogenization techniques for periodic elastic media

The aim of this section is to summarize some aspects of the asymptotic homogenization techniques as a base for successive developments.

Let us consider a body, whose reference configuration $\mathscr{B}$ can be obtained by the repetition of a small part $\mathscr{C}$, the *elementary cell*, which is made up by two or more materials (Fig. 1).

With $\varepsilon$, we denote the ratio between the characteristic lengths of $\mathscr{C}$ and $\mathscr{B}$. When $\varepsilon \downarrow 0$, the body tends to be homogeneous, for $\varepsilon$ accounts for the size of the heterogeneity.

Given a position $p \in \mathscr{B}$, let us call $x$ and $y$ its representation in two different coordinate systems, so that

$$y = \varepsilon^{-1}x. \tag{1}$$

Following the two-scales asymptotic expansion method (Bogoliubov and Mitropolsky, 1961; Nayfeh, 1973) we assume that, given a boundary value problem over $\mathscr{B}$, we look for its solution by putting any field $f$ in the form

$$f^{\varepsilon}(x) = f^{0}(x,y) + \varepsilon f^{1}(x,y) + o(\varepsilon). \tag{2}$$

Moreover, we introduce the assumption, which is crucial for the homogenization methods, that *all the fields on* $\mathscr{C}$ are periodic.

In view of Eqs. (1) and (2), it can be easily seen that

$$\operatorname{grad} f^{\varepsilon} = \operatorname{grad}_x f + \varepsilon^{-1} \operatorname{grad}_y f. \tag{3}$$

Let us assume now that the material points making up the cell are all linearly elastic, although possibly with different properties. Then, the field equations assume the form

$$\operatorname{div} \mathbf{T}^{\varepsilon} + \mathbf{b} = \mathbf{0}, \tag{4}$$

$$\mathbf{E}^{\varepsilon} = \operatorname{sym} \operatorname{grad} \mathbf{u}^{\varepsilon}, \tag{5}$$

$$\mathbf{T}^{\varepsilon} = \mathbb{A}\mathbf{E}^{\varepsilon}, \tag{6}$$

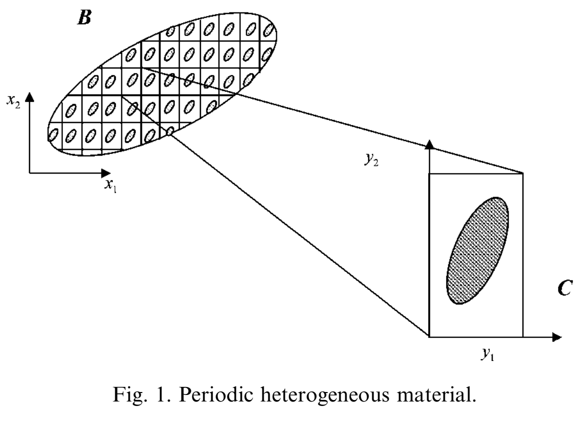


Fig. 1. Periodic heterogeneous material.

where ‘sym grad’ stands for the symmetric part of the gradient, $\mathbf{u}$ for the displacement field, $\mathbf{T}$ and $\mathbf{E}$ for the stress and strain fields, respectively, $\mathbb{A}$ for the elasticity tensor and $\mathbf{b}$ for the body-force density.

Eqs. (4)–(6) can be given an $\varepsilon$-expansion, leading to a sequence of field problems. The first two are associated with the $\varepsilon^{-1}$ and $\varepsilon^0$ terms of the expansion for the balance Eq. (4) and read

$$\mathrm{div}_y\, \mathbf{T}^0(x,y) = \mathbf{0}, \tag{7}$$

$$\mathbf{E}^0(x,y) = \mathbf{E}_x[\mathbf{u}^0(x)] + \mathbf{E}_y[\mathbf{u}^1(x,y)], \tag{8}$$

$$\mathbf{T}^0(x,y) = \mathbb{A}(y)\mathbf{E}^0(x,y), \tag{9}$$

$$-\mathrm{div}_x\, \widetilde{\mathbf{T}}^0(x) = \mathbf{b}(x), \tag{10}$$

$$\widetilde{\mathbf{E}}^0(x) = \mathbf{E}_x[\mathbf{u}^0(x)], \tag{11}$$

$$\widetilde{\mathbf{T}}^0(x) = \mathbb{A}^H \widetilde{\mathbf{E}}^0(x), \tag{12}$$

respectively, where a superposed tilde denotes average over the cell. Note that in the preceding Eqs. (8), (11), $\mathbf{u}^0(x)$ does not depend on $y$ as we require grad $\mathbf{u}$ to be bounded for $\varepsilon \downarrow 0$.

By assuming

$$\mathbf{u}^1(x,y) = \left\{\mathrm{E}_x\left[\mathbf{u}^0(x)\right]\right\}_{kl}\mathbf{w}^{kl}(y). \tag{13}$$

Eqs. (7)–(9) lead to a problem on cell $\mathscr{C}$ which will be called *microscopic*. It must be complemented by continuity conditions on the interfaces and periodic boundary conditions on $\partial\mathscr{C}$.

The field problem defined by Eqs. (10)–(12) will be called *macroscopic*, as it involves the sole variable $x$. The homogenized elasticity tensor can be shown to be

$$\mathbb{A}^H_{ijkl} = \frac{1}{|\mathscr{C}|}\int_{\mathscr{C}} \mathbb{A}_{ijmh}\left\{\delta_{km}\delta_{lh} + \left[\mathrm{E}_y(\mathbf{w}^{mh})\right]_{kl}\right\}\mathrm{d}y \tag{14}$$

(Bensoussan et al., 1978; Sanchez-Hubert and Sanchez-Palencia, 1992). This problem can be solved by employing the boundary conditions of the original problem.

## 3. Homogenization techniques for an elastic matrix with rigid inclusions

Let us now consider a linear elastic medium with rigid inclusions regularly arranged in it. It can be homogenized following the procedure described in Section 2.

In this case, the microscopic problem is formulated in terms of two sets of field equations. The behaviour of the elastic parts of the cell is described by Eqs. (7)–(9) while in the rigid parts the constitutive relation disappears, the balance equation does not change, and the strain displacement relation must fulfill the requirement $\mathbf{E}^0 = \mathbf{0}$. Making use of Eq. (13), the governing equations become

$$\mathrm{div}_y \mathbf{T}^0 = \mathbf{0}, \tag{15}$$

$$\mathbf{E}_y(\mathbf{w}^{st}) + \mathbf{E}_y(\mathbf{w}^{ts}) + 2\,\mathrm{sym}\,(\mathbf{b}_s \otimes \mathbf{b}_t) = \mathbf{0}, \tag{16}$$

where $(\mathbf{b}_i)$ is an orthonormal basis.

Eq. (16) can be solved and then used as a boundary condition on the interfaces for the field problem concerning the elastic parts. As a result, the tractions along the boundary of the rigid parts are known, thus permitting the calculation of the average stress. Under these circumstances, the calculation of the homogenized elastic coefficients is a tricky task, and can be solved in at least two ways described in details in Lévy (1987).

## 4. Rigidification techniques

The theory that a rigid body can be endowed with constitutive relations was implemented by Grioli (1983), who defined a rigid body as the limit of an elastic one.

He proved that a constrained material thus obtained is still undeformable, but has constitutive relations based on the elastic behaviour of the original body.

In the present work, we will follow the said theory as restricted to static problems and linear isotropic bodies. The rigid body is obtained by making the Young modulus tend to infinity, that is

$$\lim_{\alpha \downarrow 0} \mathbf{E} = \mathbf{0}, \tag{17}$$

where $\alpha$ stands for the inverse of the Young modulus.

We will assume that the displacement field can be described as

$$\mathbf{u}^{\alpha}(x) = \mathbf{u}^{0}(x) + \alpha \mathbf{u}^{1}(x) + o(\alpha), \tag{18}$$

where $\mathbf{u}^0$ denotes a rigid (linearized) displacement field, i.e. $\mathbf{u}^0(p) = \mathbf{u}^0(q) + \boldsymbol{\omega} \times (p - q)$. The strain field can then be described as

$$\mathbf{E}^{\alpha}(x) = \alpha \mathbf{E}^{1}(x) + o(\alpha). \tag{19}$$

Besides, we assume that the stress field can be given the asymptotic expansion

$$\mathbf{T}^{\alpha}(x) = \mathbf{T}^{0}(x) + \alpha \mathbf{T}^{1}(x) + o(\alpha), \tag{20}$$

where $\mathbf{T}^0(x)$ stands for a stress field in the rigid body and, following Marzano and Podio-Guidugli (1984), the linearized elasticity tensor is given as

$$\mathbb{A} = \alpha^{-1}\mathbb{A}_{-1} + \mathbb{A}_0. \tag{21}$$

As a consequence, the first term in the $\alpha$-expansion for the balance equation, i.e. the coefficient of $\alpha^0$, gives rise to the following field equations

$$\operatorname{div} \mathbf{T}^{0} + \mathbf{b} = \mathbf{0}, \tag{22}$$

$$\mathbf{T}^{0} = \mathbb{A}_{-1}\mathbf{E}^{1}, \tag{23}$$

$$\mathbf{E}^{1} = \text{sym grad } \mathbf{u}^{1}, \tag{24}$$

which, together with the boundary conditions on the tractions, give the solution in terms of $\mathbf{u}^1, \mathbf{T}^0$, ignoring any rigid displacement which would be immaterial. Note that the incipient strain $\mathbf{E}^1$ has no other role but that of defining the stress in a rigid body.

## 5. Homogenization and rigidification

Let us now come back to the case analysed in Section 2, assuming however that the inclusions are modelled as rigid bodies in the sense specified in Section 4, and assume that any field $f$ can be given the following double expansion

$$f^{\varepsilon\alpha}(x) = f^{00}(x,y) + f^{01}(x,y)\alpha + o(\alpha) + \left[f^{10}(x,y) + f^{11}(x,y)\alpha + o(\alpha)\right]\varepsilon + o(\varepsilon), \tag{25}$$

both for the matrix and the inclusion. The displacement field takes the form

$$\mathbf{u}^{\varepsilon\alpha}(x) = \mathbf{u}^{00}(x,y) + \mathbf{u}^{01}(x,y)\alpha + o(\alpha) + \left[\mathbf{u}^{10}(x,y) + \mathbf{u}^{11}(x,y)\alpha + o(\alpha)\right]\varepsilon + o(\varepsilon), \tag{26}$$

$$\operatorname{grad}\mathbf{u}^{\varepsilon\alpha} = [\operatorname{grad}_y\mathbf{u}^{00} + \operatorname{grad}_y\mathbf{u}^{01}\alpha + o(\alpha)]\varepsilon^{-1} + \operatorname{grad}_x\mathbf{u}^{00} + \operatorname{grad}_y\mathbf{u}^{10} + [\operatorname{grad}_x\mathbf{u}^{01} + \operatorname{grad}_y\mathbf{u}^{11}]\alpha + o(\alpha) + [\operatorname{grad}_x\mathbf{u}^{10} + \operatorname{grad}_x\mathbf{u}^{11}\alpha + o(\alpha)]\varepsilon + o(\varepsilon). \tag{27}$$

If we require $\operatorname{grad}\mathbf{u}^{\varepsilon\alpha}$ to be bounded for $\varepsilon \downarrow 0$, then

$$\operatorname{grad}_y\mathbf{u}^{00} + \operatorname{grad}_y\mathbf{u}^{01}\alpha + o(\alpha) = \mathbf{0} \quad \forall\alpha \in \mathbb{R}^+, \tag{28}$$

i.e., $\mathbf{u}^{0\alpha} = \mathbf{u}^{0\alpha}(x)$. The first two terms in the expansion for the strain field are then

$$\mathbf{E}^{00} = \mathbf{E}_x(\mathbf{u}^{00}) + \mathbf{E}_y(\mathbf{u}^{10}),$$
$$\mathbf{E}^{01} = \mathbf{E}_x(\mathbf{u}^{01}) + \mathbf{E}_y(\mathbf{u}^{11}). \tag{29}$$

In view of Eq. (23), the stress in the inclusion is given by

$$\mathbf{T}^{\varepsilon\alpha} = (\alpha^{-1}\mathbb{A}_{-1} + \mathbb{A}_0)\mathbf{E}(\mathbf{u}^{\varepsilon\alpha}), \tag{30}$$

and as a consequence

$$\mathbf{T}^{0\alpha} = \alpha^{-1}\mathbb{A}_{-1}[\mathbf{E}_x(\mathbf{u}^{00}) + \mathbf{E}_y(\mathbf{u}^{10})] + \mathbb{A}_{-1}[\mathbf{E}_x(\mathbf{u}^{01}) + \mathbf{E}_y(\mathbf{u}^{11})] + \mathbb{A}_0[\mathbf{E}_x(\mathbf{u}^{00}) + \mathbf{E}_y(\mathbf{u}^{10})] + o(\alpha). \tag{31}$$

If we assume that $\mathbb{A}_{-1}$ is nonsingular and the stress is bounded for $\alpha \downarrow 0$, in any inclusion we cannot have but $\mathbf{E}^{00} = \mathbf{0}$, thus obtaining

$$\mathbf{T}^{00} = \mathbb{A}_{-1}[\mathbf{E}_x(\mathbf{u}^{01}) + \mathbf{E}_y(\mathbf{u}^{11})] = \mathbb{A}_{-1}\mathbf{E}^{01}. \tag{32}$$

The field equations that govern the *microscopic* behaviour up to the order $\alpha^0$ in the strain and stress fields are of two kinds, one to be used for the inclusion and the other for the matrix.

The equations,

$$\operatorname{div}_y\mathbf{T}^{00} = \mathbf{0}, \tag{33}$$
$$\mathbf{E}_x(\mathbf{u}^{00}) + \mathbf{E}_y(\mathbf{u}^{10}) = \mathbf{0}, \tag{34}$$

together with

$$\mathbf{T}^{00} = \mathbb{A}_{-1}[\mathbf{E}_x(\mathbf{u}^{01}) + \mathbf{E}_y(\mathbf{u}^{11})] \tag{35}$$

are for the inclusions. Note that the constitutive function for $\mathbf{T}^{00}$ comprises only the terms of the displacement field of order $\alpha^1$, which does not match with the ones in Eq. (34).

For the matrix, the equations are

$$\operatorname{div}_y\mathbf{T}^{00} = \mathbf{0}, \tag{36}$$
$$\mathbf{E}^{00} = \mathbf{E}_x(\mathbf{u}^{00}) + \mathbf{E}_y(\mathbf{u}^{10}), \tag{37}$$
$$\mathbf{T}^{00} = \mathbb{A}[\mathbf{E}_x(\mathbf{u}^{00}) + \mathbf{E}_y(\mathbf{u}^{10})]. \tag{38}$$

Eqs. (33), (34) and (36)–(38) with such boundary conditions as: (i) continuity of the displacement on the interface and (ii) periodicity of the displacement and stress fields on the boundary of the cell, correspond to a homogenization problem with a rigid inclusion, discussed in Section 3. Its solution leads to the identification of a fictitious homogeneous material and to a *macroscopic* boundary value problem whose solution determines the field $\mathbf{u}^{00}$.

Once this problem is solved, the interface tractions are known and the problem faced is

$$\operatorname{div}_y\mathbf{T}^{00} = \mathbf{0},$$
$$m\mathbf{T}^{00} = \mathbb{A}_{-1}[\mathbf{E}_x(\mathbf{u}^{01}) + \mathbf{E}_y(\mathbf{u}^{11})] \tag{39}$$

with the tractions on the boundary obtained by the solution of the previous problem. By putting, as usual,

$$\mathbf{u}^{11} = [\mathrm{E}_x(\mathbf{u}^{01})]_{kl}\mathbf{v}^{kl}, \tag{40}$$

one obtains

$$\mathbf{E}_y(\mathbf{u}^{11}) = [\mathrm{E}_x(\mathbf{u}^{01})]_{kl}\mathbf{E}_y(\mathbf{v}^{kl}), \tag{41}$$

$$\mathrm{T}_{ij}^{00} = \mathbb{A}_{-1ijkl}[\delta_{ks}\,\delta_{lt} + [\mathrm{E}_y(\mathbf{v}^{st})]_{kl}][\mathrm{E}_x(\mathbf{u}^{01})]_{st}. \tag{42}$$

The given tractions on the boundary are

$$\mathrm{T}_{ij}^{00} = \mathbb{A}_{ijkl}[\delta_{ks}\,\delta_{lt} + [\mathrm{E}_y(\mathbf{w}^{st})]_{kl}][\mathrm{E}_x(\mathbf{u}^{00})]_{st} \tag{43}$$

and, as the traction field produced by the stress tensors (42) and (43) are the same for all $x$ on the interface, the consequence is

$$\mathbf{u}^{01} = \mathbf{u}^{00}. \tag{44}$$

Eq. (39) can now be written in terms of the vector fields $\mathbf{v}^{kl}$ which stay for the only unknowns. Their solution determines the stress field in the rigid body.

## 6. A masonry panel set across on an opening

In this section, the behaviour of a masonry panel made up by a regular texture of blocks with mortar interposed put on an opening (Fig. 2a) is analysed, in order to give an account of the possible applications of the theory. The panel is modelled as a 2D continuum, the constraints and the loads are chosen following Villaggio (1981).

The reference configuration of the panel, as well as a detail of the texture, are shown in Fig. 2b while the cell and the vectors of the translation groups generating the reference configuration, are in Fig. 3a. It is worth noting that the cell chosen here appears in a number of other works (Anthoine, 1995; Maier et al., 1991; Pande et al., 1989).

The problem is solved first by assuming the mortar as a linear elastic isotropic body and the bricks as *rigid in the limit* bodies. Here follows the description of how the described theory has been implemented.

As for the rigid parts of the cell, Eq. (34), written in the form (16), is solved. We then consider the Eqs. (37) and (38) conveniently rewritten by putting $\mathbf{u}^{10} = [\mathrm{E}_x(\mathbf{u}^{00})]_{kl}\ \mathbf{w}^{kl}$ (see Eq. (13)) and complemented by the

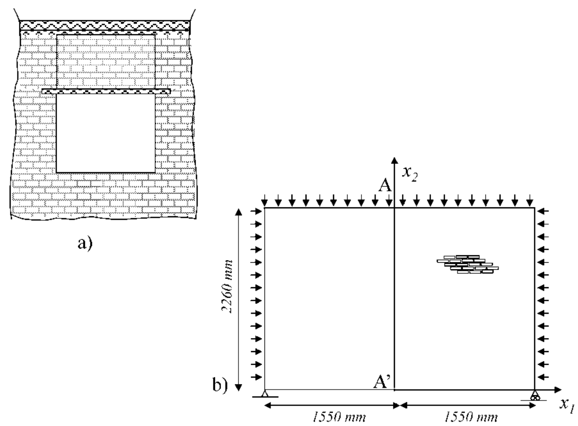


Fig. 2. The masonry panel.

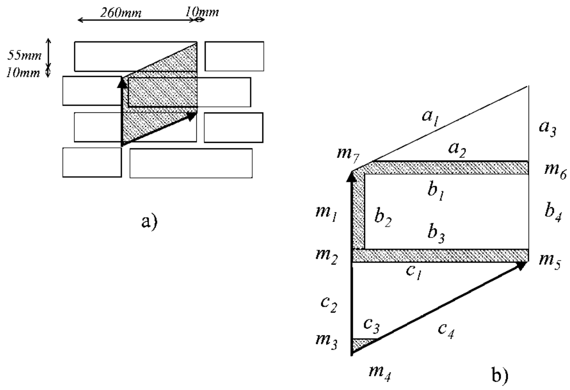


Fig. 3. The cell.

following boundary conditions (Fig. 3b): (i) along the edges $a_2, b_1, b_2, b_3, c_1, c_3$ the displacement and the traction fields are continuous; (ii) along the pairs of edges $(m_1, a_3)$, $(m_2, m_6)$, $(m_3, m_5)$ and $(m_4, m_7)$, the displacement fields are periodic and give a solution for the resulting boundary value problem.

The homogenized coefficients are then obtained by making use of the theorem of virtual power together with the periodicity conditions on the stress fields (Lévy, 1987). One could observe that the homogenized coefficients might be easily obtained by computing the average stress within the inclusions by means of the tractions along the boundary. Unfortunately, this cannot be done in the present case, as the stress is indeterminate along the edges $a_1$, $b_4$, $c_2$ and $c_4$ of the boundary.

The final step, then, is the computation of the stress within the inclusions through the solution of the field problem (39) complemented by the following boundary conditions: (i) the tractions along the edges $a_2, a_3, b_1, b_2, b_3, c_1, c_3$ are given (as the problem on the elastic parts has already been solved), (ii) the displacement and stress fields along the corresponding pairs of edges $(a_1, c_4)$ and $(b_4, c_2)$ must be continuous. The problem has been solved numerically.

Removing the assumption that the bricks are *rigid in the limit* bodies, they are instead given a linear isotropic elastic relation. The behaviour of the masonry panel considered above has then been analysed by taking into account the real arrangement of bricks and mortar, as well as using a standard homogenization technique. Numerical results have been obtained for different values of the ratio between the Young moduli of both the bricks and the mortar. A brief account of them follows.

The distribution of the normal stress component ($T_1$) along the vertical midline of the panel (line denoted by AA′ in Fig. 2b) is shown in Fig. 4.

Curves (a) and (b) are obtained using a standard homogenization technique, the former by the Young modulus in the bricks which is five times higher than in the mortar, the latter by the Young modulus 36 times higher. Curve (c) represents rigidification. Fig. 4 shows that the results obtained using the rigidification procedure are in good agreement with those obtained via a standard homogenization technique when the ratio of the Young moduli of the bricks and the mortar is 36, whereas in the other case they agree only qualitatively.

Solution (c) has been localized for a cell near the center of AA′. The results have been compared to the ones obtained for the heterogeneous model when the Young moduli ratio is 1 to 36 and shown in Fig. 5. There, curves (a) and (b) show the solutions for the heterogeneous model and for the rigidification case, respectively, and show that the predictions obtained by means of the rigidification procedure are in good agreement with those obtained for the heterogeneous model, as well.

If it is true that the homogenization techniques had already offered a simpler approach than the heterogeneous one, the present work shows that a rigidification procedure provides a further improvement.

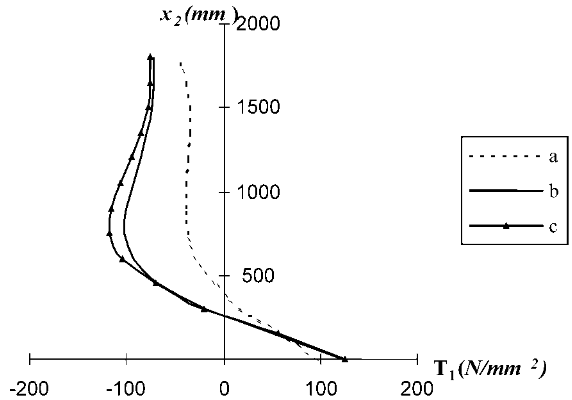


Fig. 4. Normal stress component.

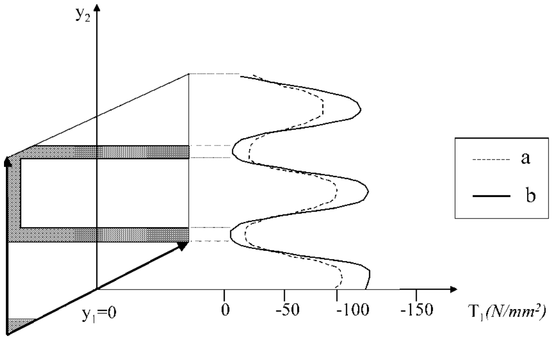


Fig. 5. Localization of the normal component.

In fact, while standard homogenization requires solving a field problem on the whole cell, rigidification solves it on a part only. Besides, the smaller the part, the bigger is the reduction of the computational effort, as it depends nonlinearly on the degrees of freedom of the problem.

## References


Anthoine, A., 1995. Derivation of the in-plane elastic characteristics of masonry through homogenization theory. International Journal of Solids and Structures 32, 137–163.

Bensoussan, A., Lions, J.L., Papanicolaou, G., 1978. Asymptotic Analysis for Periodic Structures. North-Holland, Amsterdam.

Bogoliubov, N.N., Mitropolsky, Yu., 1961. Asymptotic Methods in Non Linear Mechanics. Gordon and Breach, New York.

Grioli, G., 1983. On the stress in rigid bodies. Meccanica 28, 3–7.

Lévy, T., 1987. Fluids in porous media and suspension. In: Homogenization techniques for composite media, Lectures Notes in Physics 272, Springer, Berlin.

Maier, G., Nappi, A., Papa, E., 1991. On damage and failure of brick masonry. In: Donea, J., Jones, P.M. (Eds.), Experimental and numerical methods in earthquake engineering. Ispra, 223–245.

Marzano, S., Podio-Guidugli, P., 1984. Formulazioni alternative al vincolo di rigidità sulle deformazioni dei materiali elastici. In: Atti del VII congresso AIMETA, Sezione II: Meccanica dei Solidi. Trieste, 131–141.

Nayfeh, A.H., 1973. Perturbation Methods. Wiley, New York.

Pande, G.N., Liang, J.X., Middleton, J., 1989. Equivalent elastic moduli for brick masonry. Computational Geotechnics 8, 243–265.

Sanchez-Hubert, J., Sanchez-Palencia, E., 1992. Introduction aux Méthodes Asymptotiques et à l'homogénéisation. Masson, Paris.

Villaggio, P., 1981. Stress diffusion in masonry walls. Journal of Structural Mechanics 9, 439–450.